Fracture statistics of torsion in glass cylinders

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This paper has adopted a theoretical viewpoint for studying fracture statistics in round bars subjected to torsion, and for determining the cumulative probabilities of fracture using Weibull's and Kies–Kittl's specific-risk functions for materials that exhibit volume and surface brittleness. The use of the integral equations method has allowed us to obtain the specific-risk-of-fracture function and, in addition, to carry out a separation between the volume part and surface part for materials which show both types of brittleness at the same time. Diagrams of the cumulative probability of fracture for commercial glass samples are plotted as a practical application. The parameters of Kies–Kittl's functions regarding torsion as well as those of Weibull's functions regarding bending are appraised employing nomograms and minimizing the chi-square, respectively. Dispersion of the same is determined resorting to Fisher's information matrix. The different forms of the statistical functions followed by the same material in the two tests are due to form and size influences of the crack originating the fracture.

1. Introduction

The statistical mechanics of fracture proposed by Weibull [1] is now being widely used to describe the fracture behaviour of brittle materials. This field of Materials Science has experienced a rapid growth owing to the reliability requirements of modern engineering. Investigation has been directed to diverse aspects such as the foundations of the said mechanics [2, 3]; the application thereof for obtaining the cumulative probability of fracture concerning sundry materials subjected to different states of stress as for instance biaxial [4] and multiaxial [5] states, torsion [6–9], bending [10]; the evaluation of Weibull's parameters [11, 12]; the integral equations method for getting the specific-risk-of-fracture function [13, 14]. The size effect has been verified in glass subjected to compression [15], and glass-fibre behaviour has been analysed under combined states of traction and torsion [16]. This work endeavours to achieve a theoretical study of the torsion problem and of the fracture behaviour of commercial glass samples under torsion and flexure, determining, therefore, the cumulative probabilities of fracture, the parameters of the specific-risk-of-fracture function, and the dispersion of the same.

2. Statistics of fracture through torsion

2.1. Volume brittleness

The cumulative probability of fracture $F(\tau)$ for materials with volume brittleness and subjected to some uniaxial state of shear stress is as follows according to Weibull's theory

$$F(\tau) = 1 - \exp\left(-\frac{1}{V_0}\int_V \phi\,[\tau(r)]\,\mathrm{d}V\right) \quad (1)$$

where V_0 is the volume unit, V the body volume, r the position vector, τ the maximum shear-stress reached in the material before breaking, $\tau(r) \leqslant \tau$ the uniaxial stress field, and $\phi(\tau)$ the specific-risk-of-fracture function. Weibull [1] has proposed the following analytical form for this function

$$\phi(\tau) = \begin{cases} \left(\dfrac{\tau - \tau_L}{\tau_0}\right)^m & \tau \geqslant \tau_L \\ 0 & \tau < \tau_L \end{cases} \quad (2)$$

where τ_0 and m are parameters depending on the manufacturing process of the material, while τ_L is the stress under which there is no fracture. On the other hand, Kies [17] has proposed an analytical form including parameters m, τ_L and τ_S, where τ_S is the stress above which there is always fracture. That function has been modified by Kittl [18] through the introduction of a constant K that we shall call Kittl's constant while the new specific-risk function will be called Kies–Kittl's function. This function is given by the following expression

$$\phi(\tau) = \begin{cases} K\left(\dfrac{\tau - \tau_L}{\tau_S - \tau}\right)^m & \tau_L \leqslant \tau \leqslant \tau_S \\ 0 & \tau < \tau_L \\ \infty & \tau > \tau_S \end{cases} \quad (3)$$

Kittl's constant K is derived in a natural way when establishing the agreement between the experimental data and the functions theoretically obtained, and this constant will be explained in greater detail later.

In accordance with the elemental theory of torsion, the stress field, in cylindrical coordinates, for a round bar exhibiting volume brittleness and L in length and

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r in radius, may be expressed as follows

$$\tau(\varrho) = \frac{\varrho}{r}\tau \leqslant \tau = \frac{2M}{\pi r^3}$$

$$0 \leqslant \varrho \leqslant r; \quad 0 \leqslant \theta \leqslant 2\pi; \quad 0 \leqslant z \leqslant L \quad (4)$$

where M is the torsional moment acting on the bar. Rewriting of Equation 1 gives

$$\xi(\tau) = \ln\frac{1}{1 - F(\tau)} = \frac{1}{V_0}\int_V \phi\,(\tau(r))\,\mathrm{d}V \quad (5)$$

If $\phi(\tau)$ is given by Equation 2 – i.e. using the defined functions method – and considering the above Equations 4 and 5 we obtain

$$\xi(\tau) = \frac{2\pi Lr^2}{V_0(m+2)}\frac{\tau_0}{\tau}\left(1 + \frac{1}{m+1}\frac{\tau_L}{\tau}\right) \times \left(\frac{\tau - \tau_L}{\tau_0}\right)^{m+1} \quad (6)$$

If $\tau_L = 0$ in Equation 2, the consideration of Equations 4 and 5 yields

$$\xi(\tau) = \frac{2\pi Lr^2}{V_0(m+2)}\left(\frac{\tau}{\tau_0}\right)^m \quad (7)$$

Weibull's parameters m, τ_0 and τ_L may be obtained from Equation 6 inasmuch as $\xi(\tau)$ is known from the tests. If $\tau_L = 0$ a Weibull diagram may be plotted, and the parameters m and τ_0 may be obtained using Equation 7.

Now, if the specific-risk-of-fracture function is given by Equation 3 – i.e. if it is a Kies–Kittl function – then the consideration of Equations 4 and 5 yields

$$\xi(\tau) = \frac{2\pi Lr^2}{V_0}\left(\frac{\tau_L}{\tau_S}\right)^m\left(\frac{\tau_L}{\tau}\right)^2 K \times \int_1^{\tau/\tau_L}\left(\frac{\eta - 1}{1 - (\tau_L/\tau_S)\eta}\right)^m \eta\,\mathrm{d}\eta \quad (8)$$

If $\tau_L = 0$ in Equation 3, then Equations 4 and 5 give

$$\xi(\tau) = \frac{2\pi Lr^2}{V_0}\left(\frac{\tau_S}{\tau}\right)^2 K\int_0^{\tau/\tau_S}\frac{\eta^{m+1}}{(1-\eta)^m}\,\mathrm{d}\eta \quad (9)$$

The parameters m, τ_S and τ_L may be obtained from Equations 8 and 9, depending on τ_L, using numerical methods.

If the cumulative probability of fracture is expressed by Equation 8, then the mean fracture-stress is

$$\bar{\tau} = \frac{2\pi Lr^2}{V_0}\left(\frac{\tau_L}{\tau_S}\right)^m \tau_L K\int_1^{\tau_S/\tau_L}\left[\left(\frac{\eta - 1}{1 - (\tau_L/\tau_S)\eta}\right)^m - \frac{2}{\eta^2}\int_1^\eta\left(\frac{\xi - 1}{1 - (\tau_L/\tau_S)\xi}\right)^m \xi\,\mathrm{d}\xi\right] \times \exp\left[-\frac{2\pi Lr^2}{V_0}\left(\frac{\tau_L}{\tau_S}\right)^m\frac{K}{\eta^2} \times \int_1^\eta\left(\frac{\xi - 1}{1 - (\tau_L/\tau_S)\xi}\right)^m \xi\,\mathrm{d}\xi\right]\mathrm{d}\eta \quad (10)$$

If some known analytical form is not assumed for $\phi(\tau)$ – i.e. if using the integral equations method – then Equations 4 and 5 allow us to obtain the following integral equation where ϕ is the unknown function

$$\xi(\tau) = \frac{2\pi Lr^2}{V_0}\frac{1}{\tau^2}\int_0^\tau \phi(\eta)\,\eta\,\mathrm{d}\eta \quad (11)$$

This equation may be solved through simple derivation, and its solution is

$$\phi(\tau) = \frac{V_0}{2\pi Lr^2}\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau^2\xi(\tau)) \quad (12)$$

and the application of numerical methods to function $\xi(\tau)$ allows the evaluation of the function $\phi(\tau)$.

2.2. Surface brittleness

In the case of materials with surface brittleness, according to Weibull's theory the cumulative probability of fracture $F(\tau)$ is given by an expression similar to Equation 1 and that may be written in the following way, as Equation 5

$$\xi(\tau) = \ln\frac{1}{1 - F(\tau)} = \frac{1}{S_0}\int_S \phi[\tau(r)]\,\mathrm{d}S \quad (13)$$

where S_0 is surface unit and S is the surface of the material.

In accordance with the elemental theory of torsion, the stress field for a round bar exhibiting surface brittleness and L in length and r in radius may be expressed as follows

$$\tau(\varrho = r) = \tau$$

$$0 \leqslant \theta \leqslant 2\pi \quad 0 \leqslant z \leqslant L \quad (14)$$

If $\phi(\tau)$ is given by Weibull's function, namely Equation 2, then the consideration of Equations 13 and 14 yields

$$\xi(\tau) = \frac{2\pi Lr}{S_0}\left(\frac{\tau - \tau_L}{\tau_0}\right)^m \quad (15)$$

If $\tau_L = 0$ in Equation 2, then Equations 13 and 14 allow us to obtain

$$\xi(\tau) = \frac{2\pi Lr}{S_0}\left(\frac{\tau}{\tau_0}\right)^m \quad (16)$$

Equations 15 and 16 permit the plotting of the respective Weibull diagrams for obtaining the paramers m, τ_0 and τ_L, when $\tau_L \neq 0$ and $\tau_L = 0$, respectively.

If the specific-risk function is a Kies–Kittl function corresponding to Equation 3, then the consideration of Equations 13 and 14 yields

$$\xi(\tau) = \frac{2\pi Lr}{S_0}K\left(\frac{\tau - \tau_L}{\tau_S - \tau}\right)^m \quad (17)$$

and this Equation 17 allows us to obtain the parameters m, τ_L and τ_S.

When no known analytical form is assumed for $\phi(\tau)$, then the use of Equations 13 and 14 allow us to write the following integral equation where ϕ is the unknown function

$$\xi(\tau) = \frac{Lr}{S_0}\int_0^{2\pi}\phi(\tau)\,\mathrm{d}\theta = \frac{2\pi Lr}{S_0}\phi(\tau) \quad (18)$$

This equation may be solved in a trivial fashion, and its solution is

$$\phi(\tau) = \frac{S_0}{2\pi Lr}\,\xi(\tau) \qquad (19)$$

with which the evaluation of $\phi(\tau)$ is possible, as $\xi(\tau)$ is known from practical experience.

2.3. Combined volume and surface brittlenesses

In the case of torsion applied to some round bar presenting volume and surface brittlenesses, the respective specific-risk-of-fracture functions may be obtained in a separate manner. Considering Equations 11 and 18 we have

$$\begin{aligned} \xi(\tau) &= \xi_V(\tau) + \xi_S(\tau) \\ \xi_V(\tau) &= \frac{2\pi Lr^2}{V_0}\frac{1}{\tau^2}\int_0^\tau \phi_V(\eta)\eta\,d\eta \\ \xi_S(\tau) &= \frac{2\pi Lr}{S_0}\,\phi_S(\tau) \end{aligned} \qquad (20)$$

and if we take two groups of samples with L_i and r_i, where $i = 1, 2$, then we obtain the following equations

$$\begin{aligned} \xi_1(\tau) &= \frac{2\pi L_1 r_1^2}{V_0}\frac{1}{\tau^2}\int_0^\tau \phi_V(\eta)\eta\,d\eta + \frac{2\pi L_1 r_1}{S_0}\,\phi_S(\tau) \\ \xi_2(\tau) &= \frac{2\pi L_2 r_2^2}{V_0}\frac{1}{\tau^2}\int_0^\tau \phi_V(\eta)\eta\,d\eta + \frac{2\pi L_2 r_2}{S_0}\,\phi_S(\tau) \end{aligned} \qquad (21)$$

This system has a non-trivial solution, because if $r_1 \neq r_2$ the associated determinant is not null, and the system is linearly independent. Hence the resolution of Equation 21 for getting ϕ_V and ϕ_S yields

$$\begin{aligned} \phi_V(\tau) &= \frac{V_0}{2\pi(r_2 - r_1)}\frac{1}{\tau}\frac{d}{d\tau}\left[\left(\frac{\xi_2(\tau)}{L_2 r_2} - \frac{\xi_1(\tau)}{L_1 r_1}\right)\tau^2\right] \\ \phi_S(\tau) &= \frac{S_0}{2\pi(r_1 - r_2)}\left(\frac{r_1}{L_2 r_2}\,\xi_2(\tau) - \frac{r_2}{L_1 r_1}\,\xi_1(\tau)\right) \end{aligned} \qquad (22)$$

In this way it has been possible to separate both functions of the specific risk of fracture, when the material is exhibiting volume brittleness and surface brittleness at the same time.

3. Experimental methods

98 samples of commercial glass $r = 0.002$ m in radius and $L = 0.300$ m long were used. Half of these samples was subjected to a fracture test through torsion, and the other half was tested through flexure. The torsion arrangement included a loading disc $R = 0.07$ m in radius, a goniometer and a receptacle hanging on the disc, and this arrangement was mounted on a machine-tool lathe. Specimen holders made of bronze were bonded, by means of epoxy resin, to the ends of the specimens. This arrangement allowed us to axially insert one of the specimen holders in tailstock's centre sleeve of the lathe while the other specimen holder was inserted in axial fashion in the loading disc, thus avoiding damage to the specimens during the test. The load P of fracture through torsion was applied by introducing a number of small weights in the said loading receptacle. Upon specimen fracturing these weights were totalized in order to ascertain P. Then the maximum stress of torsional fracture is given by the following expression

$$\tau = \frac{2PR}{\pi r^3} \qquad (23)$$

The goniometer was used to verify a linear elastic behaviour until fracturing, plotting a graph not shown herein. The flexure test was a three-point bending test with the load P was concentrated at the centre of the test span $L = 0.104$ m. The maximum stress of flexural fracture is given by the following expression

$$\sigma = \frac{PL}{\pi r^3} \qquad (24)$$

The experimental results were plotted in a diagram of the cumulative probability of fracture, for both tests. The said probability was determined using the following formula

$$F(\tau) = \frac{i - 1/2}{N} \qquad (25)$$

where $F(\tau)$ is the cumulative probability of fracture, i is the number of samples that failed under some stress at most equal to τ, and N is the number of the samples tested.

4. Analysis of the results, and discussion

Fig. 1 shows diagrams of the cumulative probability of fracture for the samples tested through torsion and through flexure. First, let us consider the curve corresponding to torsion, located at the left-hand side of the graph. The experimental data were distributed in accordance with a Kies–Kittl function of the specific risk of fracture, that is to say a function given by Equation 3. In order to ascertain whether these glass samples followed this function independently of the test carried out, the flexural test was also conducted. Thus the curve shown at the right on Fig. 1 corresponds to the experimental data supplied by the three-point bending test. These data followed a distribution in keeping with a Weibull function of the specific risk of fracture, that is to say a function given by Equation 2. This fact, namely the different fracture statistics followed by the same material in the torsional and flexural tests may be explained as follows: in the instance of torsion, fracturing is through shearing and the form and size of the crack originating the fracture is of reduced influence on this process and hence the maximum stress has a low upper limit, i.e. there exists a stress τ_S. On the other hand, in the instance of flexure, fracturing is through traction; then the form of the crack originating the fracture is very important and for small cracks the maximum stress at fracture increases in value; hence such stresses have a very large upper boundary. The reason for the difference in fracture statistics between torsion and flexure can not be due to the fact that the torsion experimental configuration would introduce stresses other than shear stresses. If so these stresses would be flexure stresses

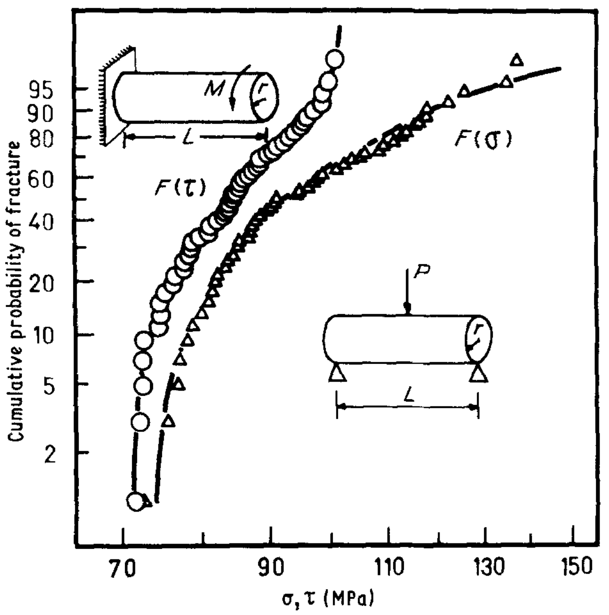

Figure 1 Diagram of cumulative probability of fracture. ○, torsion test, △, three-point bending test, of commercial glass cylinders.

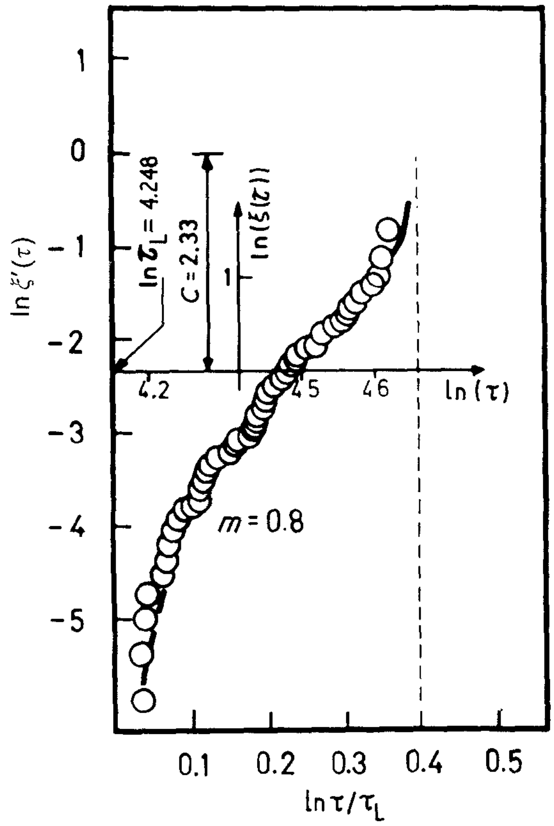

Figure 2 Kies–Kittl function's nomogram for a variable stress-field in the case of a torsion test. This nomogram allows us to obtain the parameter m and Kittl's constant K.

and in the event of becoming large then the experimental curve of torsion should exhibit some shape similar to that of the experimental curve of flexure, excepting a certain shifting due to size effect. However, experimental findings are clearly showing that such an introduction of flexure stresses is null or negligible. If the material included spherical pores instead of cracks, then fracture statistics followed would be a Weibullian one, without regard to fracturing through shearing or through traction. Fig. 1 shows that the maximum stresses of torsional fracture have upper and lower boundaries while in the case of flexure there is only a lower boundary.

The parameters of the Kies–Kittl function of the specific risk of fracture through torsion were evaluated by preparing a non-dimensional nomogram. Rearranging Equation 8, we may write

$$\ln \xi(\tau) = C + \ln \xi'(\tau)$$

$$\ln \xi'(\tau) = \ln\left[\frac{1}{(\tau/\tau_L)^2}\int_1^{\tau/\tau_L}\left(\frac{\eta - 1}{1 - (\tau_L/\tau_S)\eta}\right)^m \eta \, d\eta\right]$$

$$C = \ln\left[\frac{2\pi L r^2}{V_0}\left(\frac{\tau_L}{\tau_S}\right)^m K\right] \quad (26)$$

Now, if we plot $\ln \xi'(\tau)$ against $\ln(\tau/\tau_L)$ for several values of τ_L/τ_S and m we obtain a non-dimensional diagram. This was used to determine the respective values of parameter m and constant K. In the above group of Equations 26 and in the construction of the nomogram is to be found the justification for the existence of the constant K that is to multiply the specific-risk-of-fracture function proposed by Kies. The value of constant C is graphically ascertained by measuring the vertical distance between the horizontal axis of the nomogram and the horizontal axis of the distribution of the experimental points (see Fig. 2). Hence, in order that such a value of C be equal to the value calculated in accordance with Equation 26, it is necessary to introduce the constant K, whose value is 2.00×10^6 when the dimensions of the specimens are measured in metres and the stresses are expressed in MPa.

The parameters of the Weibull function of the specific risk of fracture through flexure were evaluated using the method of the minimum chi-square [12]. The minimum chi-squares are asymptotically efficient and squared error-consistent estimators under quite general conditions. χ^2 is given by

$$\chi^2 = \sum_{i=1}^{r}\frac{(k_i - tF_i)^2}{tF_i} \qquad \sum_{i=1}^{r} k_i = t \quad (27)$$

where the population is classified into r classes each

TABLE I Statistical parameters of the Kies–Kittl distribution (torsion) and of the Weibull distribution (flexure)

Torsion			Flexure		
Nomogram	$\bar{\tau}$	χ^2	minimum χ^2	$\bar{\sigma}$	χ^2
$m = 0.8$			$m = 0.5$		
$\tau_L = 70.0$ MPa	85.1 MPa	1.952	$\sigma_L = 65$ MPa	97.1 MPa	4.648
$\tau_S = 104.0$ MPa			$\sigma_0 = 5.5 \times 10^{-3}$ MPa		
$\chi^2_{0.95,1} = 3.84$			$\chi^2_{0.95,2} = 5.99$		

comprising k_i elements, t is the number of trials and F_i is the probability of failure in the classes. The cumulative probability of fracture for a round cylinder subjected to the three-point bending test is given by the following expression [10]

$$F(\sigma) = 1 - \exp\left[-\frac{2Lr^2}{V_0(m+1)}\left(\frac{\sigma_L}{\sigma_0}\right)^m \frac{1}{\sigma/\sigma_L} \times \int_1^{\sigma/\sigma_L} \frac{(\eta-1)^{m+1}}{\eta}\left(1-\frac{\eta^2}{(\sigma/\sigma_L)^2}\right)^{1/2} d\eta\right] \quad (28)$$

where m and σ_0 are parameters depending on the manufacturing process of the material, and σ_L is the lower-limit stress under which there is no fracture. In view of the complex structure of the above Equation 28 when deriving Equation 27 with respect to each one of the three parameters, the solution of the system to be solved becomes complicated. Hence it is necessary to programme an algorithm with backtracking in order to reach the minimum chi-square using the general Equation 27. The backtracking algorithm is very efficient for finding the minimum value of the chi-square function. One of the problems regarding this method is the case of a small sample, to which it is not applicable. Another difficulty is the choice of adequate classes having at least five elements each.

The values of the parameters were indicated in Table I, along with the mean stress at fracture and the χ^2 of the distribution of probability used, for both tests, obtained in keeping with the methods hereinabove explained.

The dispersion of the parameters of the cumulative-probability-of-fracture functions may be estimated through Fisher's information matrix [15]. The coefficients of the Fisher matrix are determined using the following relationship:

$$r_{ij} = -n\int\left(\frac{\partial^2 \ln f(\tau;\theta)}{\partial\theta_i\partial\theta_j}\right)f(\tau)\,d\tau \quad (29)$$

where r_{ij} is the coefficient i, j, n is sample size, θ are the parameters, and $f(\tau) = dF(\tau)/d\tau$ is the density function of fracture probability. In view of the complex structure of the function of the cumulative probability of torsional fracture, when the specific-risk function is a Kies function, with $\tau_L \neq 0$, the required calculations in the Fisher matrix are very cumbersome, and hence it becomes more convenient to use another method for obtaining the dispersion of the parameters; for instance, the Monte Carlo simulation method may be resorted to.

5. Conclusions

The problem of obtaining the specific-risk-of-fracture function $\phi(\tau)$ for the case of torsion has been completely solved using both the defined-functions method and the integral-functions method, for the two instances of volume brittleness and of surface brittleness, either separated or combined. In addition, the specific-risk-of-fracture function proposed by Kies has been subjected to a correction through the introduction of a constant K. The different fracture statistics followed by the commercial-glass samples when subjected to torsion or to bending, are due to the influence of the form and size of the crack originating the fracture. In the case of torsion (fracture through shearing), the form and size are of reduced importance, which involves the existence of an upper-limit stress τ_S, and the statistics followed were Kies–Kittl statistics. On the other hand, in the case of flexure (fracture through traction), the form and size of the crack are very important, which involves for the fracture stresses the existence of a very large upper-limit stress, and then the statistics followed are Weibull statistics. It should be underlined that the integral equations method is very important because, without assuming some known analytical form for the specific-risk-of-fracture function, the same may be obtained by applying some numerical method to the function $\xi(\tau)$ which is known from practical experience.

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